Numerical Methods
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Underdetermined problems (2h.)
(FOCUSS, M-FOCUSS, Applications)
Introduction

- Solutions to underdetermined linear systems,
- Morphological constraints,
- FOCUSS algorithm,
- M-FOCUSS algorithm.
Bibliography

Solutions to linear systems

A system of linear equations can be expressed in the following matrix form:

$$Ax = b,$$  \hspace{1cm} (1)

where $A = [a_{ij}] \in \mathbb{R}^{M \times N}$ is a coefficient matrix, $b = [b_i] \in \mathbb{R}^M$ is a data vector, and $x = [x_j] \in \mathbb{R}^N$ is a solution to be estimated. Let $B = [A \ b] = \mathbb{R}^{M \times (N+1)}$ be the augmented matrix to the system (1). The system of linear equations may behave in any one of three possible ways:

A The system has no solution if $\text{rank}(A) < \text{rank}(B)$.

B The system has a single unique solution if $\text{rank}(A) = \text{rank}(B) = N$.

C The system has infinitely many solutions if $\text{rank}(A) = \text{rank}(B) < N$. 
Solutions to linear systems

The **case C** occurs for *rank-deficient* problems or *under-determined* problems. In spite of infinitely many solutions, a good approximation to a true solution can be obtained if some *a priori* knowledge about the nature of the true solution is accessible. The additional constraints are usually concerned with a degree of sparsity or smoothness of the true solution.
RREF

Basic variables

Free variables
Example

Let:

\[
\begin{cases}
  x_1 + 3x_2 + x_3 = a \\
  -x_1 - 2x_2 + x_3 = b \\
  3x_1 + 7x_2 - x_3 = c
\end{cases}
\]

\[
\begin{bmatrix}
  1 & 3 & 1 & | & a \\
  -1 & -2 & 1 & | & b \\
  3 & 7 & -1 & | & c
\end{bmatrix}
\]

Gauss-Jordan

\[
\begin{bmatrix}
  1 & 0 & -5 & | & -2a - 3b \\
  0 & 1 & 2 & | & b + a \\
  0 & 0 & 0 & | & c - a + 2b
\end{bmatrix}
\]

The system is consistent (it has solutions) if

\[
c - a + 2b = 0 \quad \Rightarrow \quad c = a - 2b
\]

Solution: \( x_3 = \) free variable, \( x_2 = a + b - 2x_3, \) \( x_1 = -2a - 3b + 5x_3. \)
Example

Let:
\[
\begin{align*}
  x_1 + 2x_2 + 2x_3 + 3x_4 &= 4 \\
  2x_1 + 4x_2 + x_3 + 3x_4 &= 5 \\
  3x_1 + 6x_2 + x_3 + 4x_4 &= 7
\end{align*}
\]

\[
\begin{bmatrix}
  1 & 2 & 2 & 3 & 4 \\
  2 & 4 & 1 & 3 & 5 \\
  3 & 6 & 1 & 4 & 7
\end{bmatrix}
\rightarrow
\begin{bmatrix}
  1 & 2 & 0 & 1 & 2 \\
  0 & 0 & 1 & 1 & 1 \\
  0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Consistent

Solution:
\[
\begin{align*}
  x_1 &= 2 - 2x_2 + x_4, \\
  x_2 &= \text{free variable,} \\
  x_3 &= 1 - x_4, \\
  x_4 &= \text{free variable.}
\end{align*}
\]
Example

Thus:

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{bmatrix} = \begin{bmatrix}
  2 - 2x_2 - x_4 \\
  x_2 \\
  1 - x_4 \\
  x_4
\end{bmatrix} = \begin{bmatrix}
  2 \\
  0 \\
  0 \\
  0
\end{bmatrix} + x_2 \begin{bmatrix}
  -2 \\
  1 \\
  0 \\
  0
\end{bmatrix} + x_4 \begin{bmatrix}
  -1 \\
  0 \\
  0 \\
  1
\end{bmatrix}
\]

General solution  Particular solution  Homogenous solution $\in \mathbb{N}(A)$

$x_{\text{general}} = x_{\text{particular}} + x_{\text{homogeneous}}$

The homogeneous solution is a solution to the system $Ax = 0$.

Problem: How to choose free variables?  Remark: Replacing free variables with zero-values is not always a good solution!
Sparseness constraints

- Regularized least-squares problem:

\[
\min_x \left\{ \|Ax - b\|_2^2 + \gamma E^{(p)}(x) \right\}
\]

- \(L_p\) diversity measure (Gorodnitsky, Rao, 1997):

\[
E^{(p)}(x) = \text{sgn}(p) \sum_{j=1}^{N} |x_j|^p \quad p \leq 1.
\]
FOCUSS algorithm

Let 

$$J(x) = \|Ax - b\|_2^2 + \gamma E^{(p)}(x)$$

Stationary point:

$$\nabla J(x_*) = 2A^T A x_* - 2A^T b + 2\lambda W^{-2}(x_*)x_* = 0$$

$$W(x_*) = \text{diag}\left\{ |x_j|^{1-p/2} \right\}, \quad \lambda = \frac{|p|}{2\gamma}$$

Hence:

$$x_* = W^2(x_*) A^T (AW^2(x_*) A^T + \lambda I_M)^{-1} b.$$ 

Iterative updates:

$$x_{k+1} = W^2(x_k) A^T (AW^2(x_k) A^T + \lambda I_M)^{-1} b.$$
FOCUSS algorithm

\[ p \in [0,1], \quad \lambda > 0 \]
Randomly initialize \( x^{(0)} \),

**For** \( k = 0,1,2,\ldots \)** until convergence do**

\[ W_k = \text{diag}\left\{ \left| x_j^{(k)} \right|^{1-p/2} \right\}, \]

\[ x^{(k+1)} = W_k^2 A^T \left( AW_k^2 A^T + \lambda I_M \right)^{-1} b, \]
Wiener filtered FOCUSS algorithm

\[ p \in [0,1], \quad \lambda > 0 \]

Randomly initialize \( x^{(0)} \),

For \( k = 0,1,2,... \) until convergence do

\[ W_k = \text{diag} \left\{ \left| x_j^{(k)} \right|^{\frac{1-p}{2}} \right\}, \]

\[ x^{(k+1)} = W_k^2 A^T \left( A W_k^2 A^T + \lambda I_M \right)^{-1} b, \]

\[ x^{(k+1)} \leftarrow F \left( x^{(k+1)} \right) \]

Wiener filtered FOCUSS algorithm

Markov Random Field (MRF)

<table>
<thead>
<tr>
<th>NW</th>
<th>N</th>
<th>NE</th>
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</thead>
<tbody>
<tr>
<td>W</td>
<td>j</td>
<td>E</td>
</tr>
<tr>
<td>SW</td>
<td>S</td>
<td>SE</td>
</tr>
</tbody>
</table>

First and second-order interactions

\[ N_j = \begin{cases} j - h - 1, j - h, j - h + 1, \\ j - 1, j + 1, \\ j + h - 1, j + h, j + h + 1 \end{cases} \]

\[ L = 9 \]

- local mean around j-th pixel
- local variance

\[
\mu_j = \frac{1}{L-1} \sum_{n \in N_j} x_n^{(k+1)}
\]

\[
\sigma_j^2 = \frac{1}{L-1} \sum_{n \in N_j} \left( x_n^{(k+1)} \right)^2 - \mu_j^2
\]
Wiener filtered FOCUSS algorithm

• Update rule

\[ x_j^{(k+1)} \leftarrow \mu_j + \frac{\sigma_j^2 - \nu^2}{\sigma_j^2} \left( x_j^{(k+1)} - \mu_j \right) \]

• Mean noise variance

\[ \nu^2 = \frac{1}{N} \sum_{j=1}^{N} \sigma_j^2 \]
Limited-view tomographic imaging

\[ \mathbf{A}_{(2 \times 2)} = \begin{bmatrix}
1 & 1 & 0 & 0 \\
\alpha & 0 & 0 & \alpha \\
0 & \alpha & \alpha & 0 \\
0 & 0 & 1 & 1
\end{bmatrix} \quad \alpha = \frac{\sqrt{5}}{2} \]

(Rank-deficient)

\[ \text{rank} \left( \mathbf{A}_{(2 \times 2)} \right) = 3 \]

\[ \text{null}(\mathbf{A}_{(2 \times 2)}) = \text{span}\left\{ \begin{bmatrix} -1, 1, -1, 1 \end{bmatrix}^T \right\} \]

\[ LSS(\mathbf{A}; \mathbf{b}) = \{ \mathbf{x} \in \mathbb{R}^N : \| \mathbf{A}\mathbf{x} - \mathbf{b} \|_2 = \text{min}! \} \]

\[ LSS(\mathbf{A}; \mathbf{b}) = \mathbf{x}_{LS} + N(A) \quad \text{where} \quad \mathbf{x}_{LS} = \mathbf{x}_r = P_R(\mathbf{A}^T)\mathbf{x}_{exact} \]
Tomographic imaging example

Phantom image

Minimal $l_2$ norm least squares solution:
(LS algorithms: ART, SIRT)
Tomographic imaging
(noise-free data)

FOCUSS algorithm
(p = 1, k = 15, \( \lambda = 10^{-8} \))

Wiener filtered FOCUSS algorithm
(p = 1, k = 15, \( \lambda = 10^{-8} \))
Tomographic imaging
(noisy data SNR = 30 dB)

FOCUSS algorithm
($p = 1$, $k = 15$, $\lambda = 40$)

Wiener filtered FOCUSS algorithm
($p = 1$, $k = 15$, $\lambda = 40$)
Tomographic imaging
(Normalized RMSE, noise-free)
Tomographic imaging (Normalized RMSE, p = 1, noisy data)
Let \( AX + N = B \) where \( A \in \mathbb{R}^{M \times N} \), \( X = [x_1, \ldots, x_T] \in \mathbb{R}^{N \times T} \), \( B \in \mathbb{R}^{M \times T} \), \( M \leq N \), (under-determined) \( \text{rank}(A) = M \), \( T > N \), \( N = [n_1, \ldots, n_T] \in \mathbb{R}^{M \times T} \).

*Cotter, Rao, Engan, Kreutz-Delgado, 2005* (additive noise)

If \( M < N \), the nullspace of \( A \) is non-trivial, thus additional constraints are necessary to select the right solution. The **M-FOCUSS** assumes sparse solutions.

**Theorem:** The sparse solution to the consistent system \( AX = B \) is unique, if \( \text{rank}(B) = T \), with \( T \leq M \), any \( M \) columns of \( A \) are linearly independent (unique representation property (URP) condition), and for each \( t \): \( x_t \) has at most \( \left\lceil \left( (M + T) / 2 \right) - 1 \right\rceil \) nonzero entries, where \( \left\lceil \cdot \right\rceil \) is a ceil function.
The M-FOCUSS algorithm iteratively solves the following equality constrained problem:

$$\min_x J^{(p)}(X), \quad \text{s.t.} \quad AX = B,$$

where

$$J^{(p)}(X) = \sum_{j=1}^{N} \left\| x_j^T \right\|_2^p = \sum_{j=1}^{N} \left( \sum_{t=1}^{T} x_{jt}^2 \right)^{p/2}. \quad - l_p \text{ diversity measure for sparsity}$$

$$0 \leq p \leq 2 \quad - \text{degree of sparsity} \quad x_j^T \quad - \text{the } j\text{-th row of } X$$

(LS solution with minimal $l_2$-norm)

($l_0$ norm solution – NP-hard problem)
M-FOCUSS

For inconsistent data, i.e. \( B \notin \text{R}(A) \), Cotter at al. developed the regularized M-FOCUSS algorithm that solves the Tikhonov regularized least-squares problem in a single iterative step:

\[
X^{(k)} = \arg \min_X \Psi\left( X \mid X^{(k-1)} \right), \quad \text{where} \quad \Psi\left( X \mid X^{(k-1)} \right) = \|B - AX\|_F^2 + \lambda \left\| W^{-1}X \right\|_F^2,
\]

\[
W = \text{diag}\left( w_j^{1-p/2} \right), \quad \text{with} \quad w_j = \left( \sum_{t=1}^{T} (x_{jt}^{(k-1)})^2 \right)^{1/2} = \left\| (x_j^T)^{(k-1)} \right\|_2.
\]

**Regularized M-FOCUSS:**

For \( k = 1, 2, \ldots \)

\[
A^{(k+1)} = AW^{(k)},
\]

\[
X^{(k+1)} = W^{(k)}\left( A^{(k)} \right)^T \left( A^{(k)} \left( A^{(k)} \right)^T + \lambda I_M \right)^{-1} B.
\]